## Solving Quadratics by Completing the Square

If a quadratic equation is composed of a perfect square, then it is easy to solve by extracting the root. Using a technique called **completing the square,** we can transform any quadratic equation into one composed of a perfect square.

(Completing the square is not a popular method for solving quadratic equations, but the technique is useful in other applications, so discussing the method is important.)

### Completing the square

If we look at the pattern formed by the perfect square $(a+b)^2 = a^2 + 2ab + b^2$, we notice that the middle term is double the product of the square root of the end terms.

In general, we will see the perfect square as $(x+k)^2 = x^2 + 2xk + k^2$, we can see the relationship between the last two terms. $k^2 = (\frac{1}{2}(2k)^2)$ the last term is the square of one-half the coefficient of x.

Using this relationship we can change any quadratic into a perfect square.

Examples:

Change $x^2 + 10x$ into a perfect square

| | |
|---|---|
| $x^2+10x$ | ½ of 10 is 5 |
| $x^2 + 10x + 5^2 - 5^2$ | Add $5^2$ to the problem, also subtract the same thing to maintain the integrity of the problem |
| $(x^2 + 10x + 25) - 25$ | Factor the first part. |
| $(x + 5)^2 - 25$ | Note: the + comes from the +10x and the 5 is the square root of the 25 in the parenthesis |

| | |
|---|---|
| $x^2 - 2x$ | ½ of 2 is 1 |
| $x^2 - 2x + 1^2 - 1^2$ | Add $1^2$ to the problem and also subtract $1^2$ |
| $(x^2 - 2x + 1)^2 - 1$ | Factor the first part |
| $(x - 1)^2 - 1$ | Note: the – comes from the –2x and the 1 is the square root of the 1 in the parenthesis |

| | |
|---|---|
| $r^2 + 7r$ | ½ of 7 is $7/2$, you could say 3.5 but that is harder to work with |
| $r^2 + 7r + 7/2^2 - 7/2^2$ | Add $7/2^2$ to the problem and also subtract it off |
| $(r^2 + 7r + 49/4) - 49/4$ | Factor the first part |
| $(r + 7/2)^2 - 49/4$ | Note: the + is from $+7/2$ and the $7/2$ is the square root of $49/4$ |

Solving Quadratic Equations using Completing the Square

1. Write the problem with variables on one side of the equal sign and constants on the other side
2. Divide both sides of the equation by the coefficient of the squared term (completing the square requires there to be a 1 in the first term)
3. Complete the square on the variable side
4. Add the new constant to both sides (eliminates it from the variable side)
5. Solve the equation by extracting the root

Examples:

Solve $x^2 + 2x - 4 = 0$

| | |
|---|---|
| $x^2 + 2x - 4 = 0$ | Move 4 to other side |
| $x^2 + 2x = 4$ | Complete the square |
| $(x^2 + 2x - 1) - 1 = 4$ | Add 1 to both sides |
| $(x^2 + 2x - 1) = 5$ | Factor the variable side |
| $(x + 1)^2 = 5$ | Extract the root |
| $(x+1) = \sqrt{5}$ or $(x + 1) = -\sqrt{5}$ | Solve equations |
| $x = \sqrt{5} - 1$ or $x = -\sqrt{5} - 1$ | |

Solve $2y^2 - 12y - 7 = 0$

| | |
|---|---|
| $2y^2 - 12y - 7 = 0$ | Add seven to both sides |
| $2y^2 - 12y = 7$ | Divide both sides by 2 |
| $y^2 - 6y = \frac{7}{2}$ | Complete the square |
| $y^2 - 6y + 3^2 - 3^2 = \frac{7}{2}$ | |
| $y^2 - 6y + 9 - 9 = \frac{7}{2}$ | Add 9 to both sides |
| $y^2 - 6y + 9 = \frac{7}{2} + 9$ | |
| $y^2 - 6y + 9 = \frac{25}{2}$ | Factor left side |
| $(y - 3)^2 = \frac{25}{2}$ | Extract the root |
| $y - 3 = \sqrt{\frac{25}{2}}$ or $y - 3 = -\sqrt{\frac{25}{2}}$ | Add 3 to both sides |
| $y = \sqrt{\frac{25}{2}} + 3$ or $y = -\sqrt{\frac{25}{2}} + 3$ | Simplify |
| $y = \frac{5\sqrt{2}}{2} + 3$ or $y = -\frac{5\sqrt{2}}{2} + 3$ | |
| $y = \frac{6+5\sqrt{2}}{2}$ or $y = -\frac{6+5\sqrt{2}}{2}$ | |